



Sinyaller ve Sistemler

“Z-Dönüşümü”

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İçindekiler

- Z-Dönüşüm
- Z-Dönüşüm Özellikleri
- Region of Convergence (ROC)
- Block Diagram Manipulation
- Inverse Z-Transform
- Solving Difference Equations
- Transfer Function

Z-DÖNÜŞÜM

- Sürekli zamanlı sinyallerin zaman alanından frekans alanına geçişi Fourier ve Laplace dönüşümleri ile mümkün olmaktadır.
- Laplace, Fourier dönüşümünün genel bir şeklidir.
- Ayırık zamanlı sinyaller için de ayırık zamanlı Fourier dönüşümleri kullanılmaktadır.
- z dönüşüm de ayırık zamanlı Fourier dönüşümünün genel bir şeklidir.
- Ayırık zamanlı $f(n)$ sinyalinin (veya dizinin) z dönüşümü aşağıdaki gibi ifade edilir.

Basic Concepts

- Consider a sequence of values: $\{x_k : k = 0, 1, 2, \dots\}$
- These may be samples of a function $x(t)$, sampled at instants $t = kT$; thus $x_k = x(kT)$.
- The Z transform is simply a polynomial in z having the x_k as coefficients:

$$X(z) = Z\{x_k\} = \sum_{k=0}^{\infty} x_k z^{-k}$$

Z-Transform

- Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Bilateral - Unilateral

- Two sided or bilateral z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Unilateral z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Z-Dönüşümü

- Ayırık değer ve indisleri birlikte verilir.
- İndisler positif ise üsler negatif; indisler negatif ise üsler pozitif.
- İndisler sıralı alınır. ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, ...
- Olmayan indislerin çarpım katsayıları sıfır alınır. Diğer katsayılar olduğu gibi alınır.

Example of z-transform

n	$n \leq -1$	0	1	2	3	4	5	$N > 5$
$x[n]$	0	2	4	6	4	2	1	0

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

Unit Step Signal - continued

A little bit of math ...

$$\begin{aligned}1 + a + a^2 + \dots + a^n &= \frac{(1-a)(1+a+a^2+\dots+a^n)}{1-a} \\ &= \frac{1-a^{n+1}}{1-a}\end{aligned}$$

$n \rightarrow \infty$, assuming $|a| < 1$,

$$\begin{aligned}1 + a + a^2 + \dots &= \lim_{n \rightarrow \infty} \frac{(1-a)(1+a+a^2+\dots+a^n)}{1-a} \\ &= \lim_{n \rightarrow \infty} \frac{1-a^{n+1}}{1-a} \\ &= \frac{1}{1-a}\end{aligned}$$

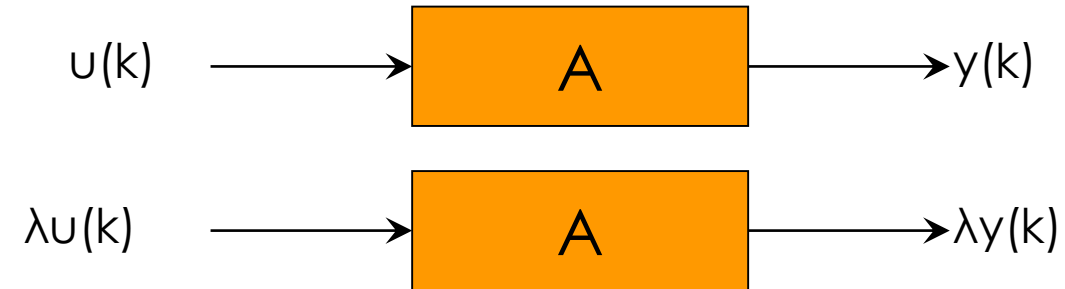
$$U_{\text{step}}(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = \frac{1}{1-z^{-1}}$$



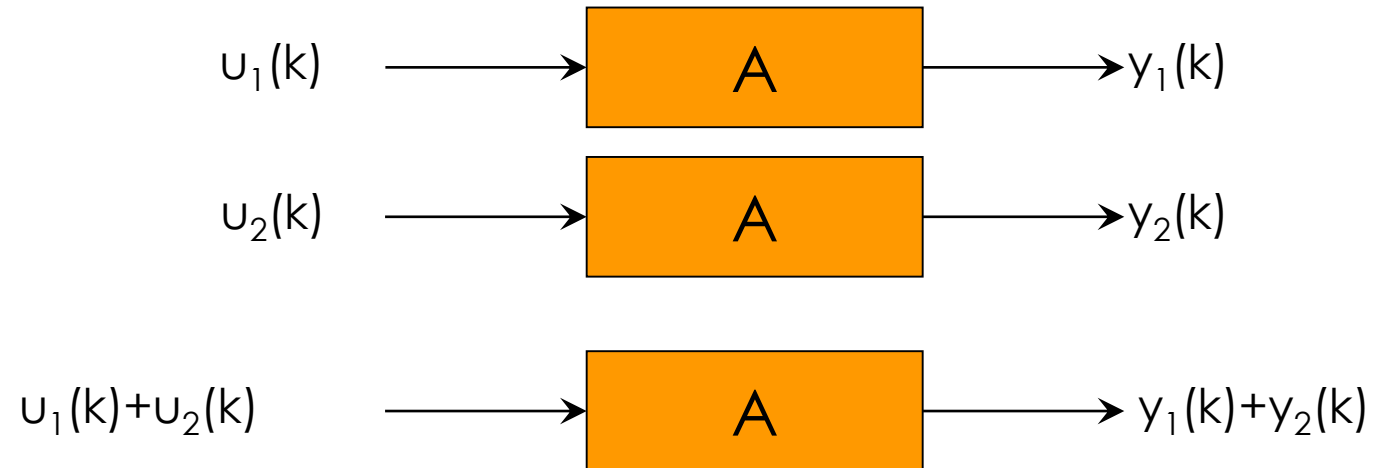
Z-Dönüşüm Özellikleri

What does “Linear” mean exactly?

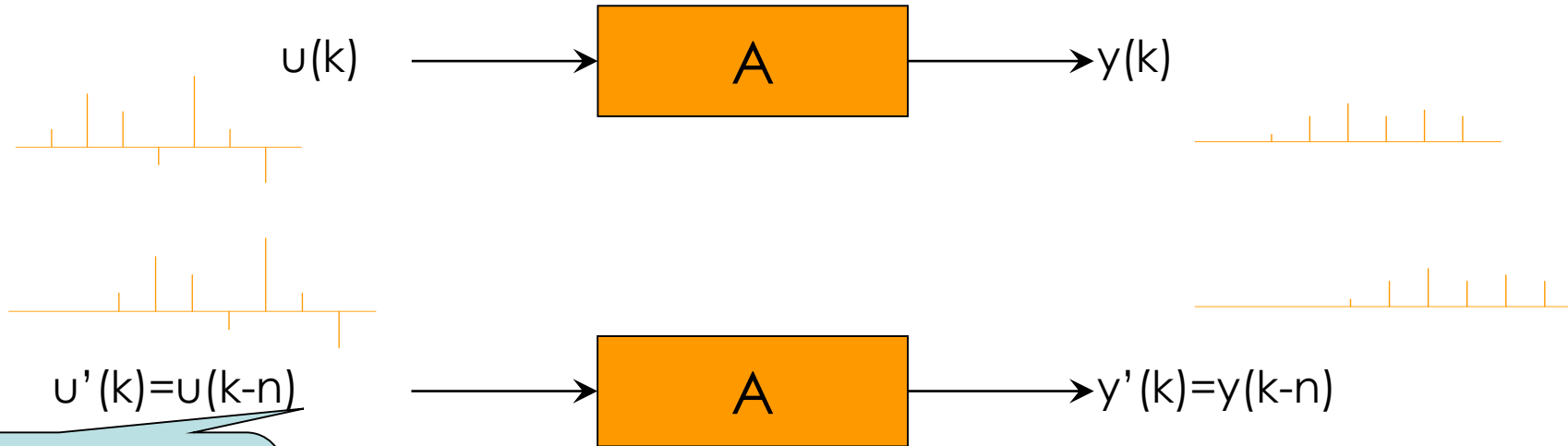
- Scaling



- Superposition



Time Invariance



Idiom:
 $u(k-n)$ is $u(k)$ delayed
by n time units!

Properties of the Z-Transform

• **Linearity:** $ax_1[n] + bx_2[n] \Leftrightarrow aX_1[z] + bX_2[z]$

Proof:

$$\mathcal{Z}\{ax_1[n] + bx_2[n]\} = \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])z^{-n} = \sum_{n=-\infty}^{\infty} ax_1[n]z^{-n} + \sum_{n=-\infty}^{\infty} bx_2[n]z^{-n} = X_1[z] + X_2[z]$$

• **Time-shift:** $x[n - n_0] \Leftrightarrow z^{-n_0} X[z]$

Proof:

$$\begin{aligned} \mathcal{Z}\{x[n - n_0]\} &= \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= \sum_{m=-\infty}^{\infty} x[m]z^{-m} z^{-n_0} = z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-n_0} X[z] \end{aligned}$$

What was the analog for CT signals and the Laplace transform?

• **Multiplication by n :** $nx[n] \Leftrightarrow -z \frac{dX[z]}{dz}$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -n \sum_{n=-\infty}^{\infty} x[n]z^{-n-1} \Rightarrow -z \frac{dX(z)}{dz} = n \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{nx[n]\}$$

Zamanda Kaydırma

İndisin 0 olduğu yerin kaydırılması

$x(n - n_0)$ sinyalinin z-dönüşümü $X(z)z^{-n_0}$ dir.

Örnek: $x(n) = \{1, 2, 5, 7, 0, 1\}$ ise

a) $x_1(n) = x(n + 2)$

b) $x_2(n) = x(n - 2)$

sinyallerinin z-dönüşümünü bulunuz.

a) $X_1(z) = X(z)z^2$

$$X(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-1}$$

$$X_1(z) = z^4 + 2z^3 + 5z^2 + 7z + z^{-1}$$

b) $X_2(z) = X(z)z^{-2}$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

Convolution

Time Domain

$u(k)$

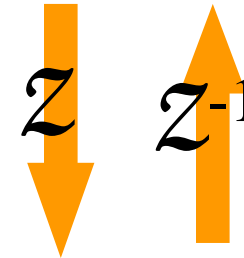
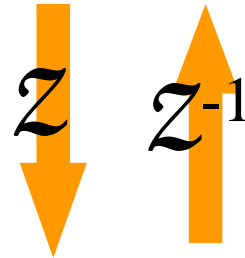
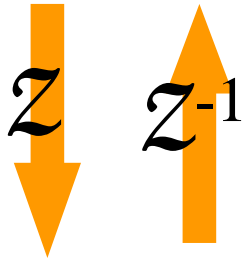
*

$v(k)$

=

$y(k)$

(convolution)



Z Domain

$U(z)$

.

$V(z)$

=

$Y(z)$

(multiplication)

Evrişim - Konvalizasyon

Örnek: Aşağıdaki ayırık zamanlı sinyallerin evrişimini bulunuz.

$$x_1(n) = \{1, -2, 1\}$$

↑
n=0

$$x_1(n) * x_2(n)$$

$$= X_1(z)X_2(z)$$

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{diğer} \end{cases}$$

Çözüm 1: $x_1(n)$ 'in z-dönüşümü

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$x_2(n)$ 'in z-dönüşümü

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) =$$

$$(1 - 2z^{-1} + z^{-2})(1 - z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$= 1 - z^{-1} + z^{-6} + z^{-7}$$

Ters z-dönüşüm alınırsa, $x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

↑

Çözüm 2: $x_1(n)$ ile $x_2(n)$ sinyallerinin evrişimi

n:	0	1	2	3	4	5	6	7
$x_2(n)$	1	1	1	1	1	1	0	0
$x_1(n)$	1	-2	1					
		1	$x(0)$					
	1	-2	1					
		-2	1	$x(1)=-1$				
		1	-2	1				
	1	-2	1	$x(2)=-0$				
		1	-2	1				
		1	-2	1	$x(3)=0$			
			1	-2	1			
			1	-2	1	$x(4)=0$		
				1	-2	1		
				1	-2	1	$x(5)=0$	
					1	-2	1	
					1	-2	0	$x(6)=-1$
						1	-2	1
						1	0	0
								$x(7)=1$

$$x_1(n) * x_2(n)$$

$$= X_1(z)X_2(z)$$



Region of Convergence (ROC)

Kutuplar ve Sıfırlar

- Bir z-dönüşümünün kutupları, o z-dönüşümünü sonsuza götüren z değerleridir.
- Bir z-dönüşümünün sıfırları, o z-dönüşümünü sıfıra götüren z değerleridir.

Örnek: $X(z) = \frac{1}{1-az^{-1}}$ fonksiyonunun kutup ve sıfırlarını bulunuz.

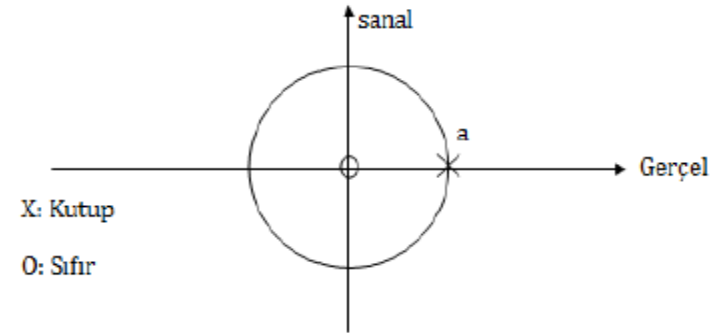
Çözüm: $X(z)$ fonksiyonunun pay ve paydasını z ile çarpalım

$$\text{---} \rightarrow X(z) = \frac{z}{z-a}$$

$$\text{Sıfır: } z = 0 \quad \text{---} \rightarrow X(z) = 0$$

$$\text{Kutup: } z = a \quad \text{---} \rightarrow X(z) = \infty$$

$X(z)$ fonksiyonunun kutup-sıfır grafiğini çiziniz.



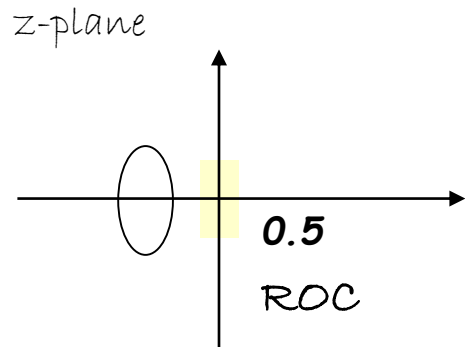
Z-Transform

- Region of Convergence (ROC)

In general, the z-transform is an infinite sum! This means it (the z-transform) may not exist for all values of z. More specifically, it is the value of $r = |z|$ that is important. If

$x(n) = (0.5)^n u(n)$, then

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} (0.5)^n z^{-n} \\ &= \sum_{n=0}^{\infty} (0.5 z^{-1})^n \\ &= \frac{1}{1 - 0.5 z^{-1}} \end{aligned}$$

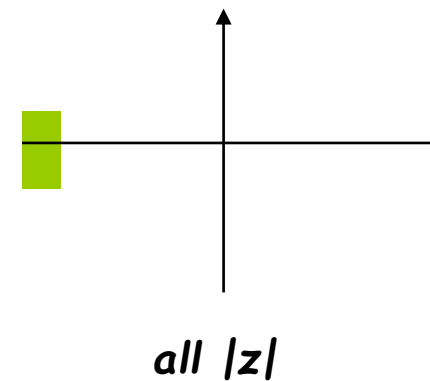
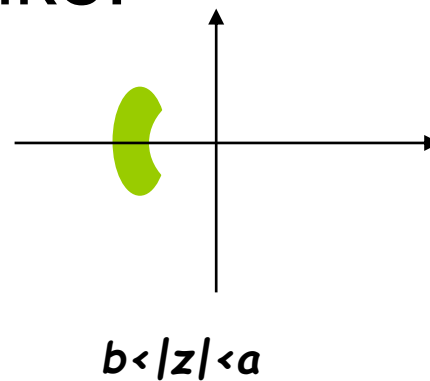
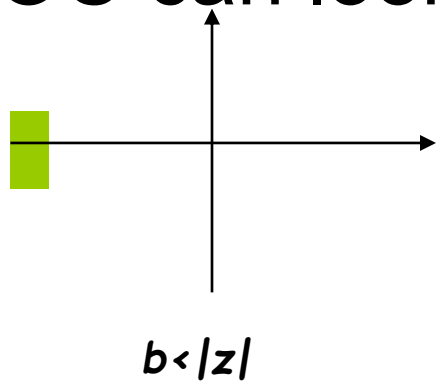
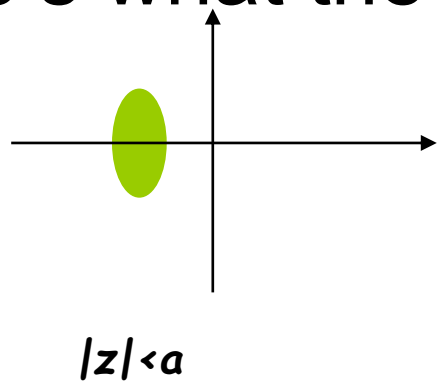


only if $|z| > 0.5!$

Z-Transform

- Region of Convergence

Here's what the ROC can look like:



Z-Transform Yakınsama Bölgesi

Örnek: Aşağıdaki dizilerin z-dönüşümünü hesaplayınız ve yakınsama bölgesini bulunuz.

$$\text{a) } x(n) = \{1, 2, 5, 7, 0, 1\}$$

\uparrow
 $n=0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = X(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

$X(z)$ $z = 0$ ve $z = \infty$ değerleri için tanımsız olmakta, diğer bütün z değerleri için sonlu olmaktadır. O halde yakınsama bölgesi, $z=0$ ve $z=\infty$ hariç bütün z düzlemidir.

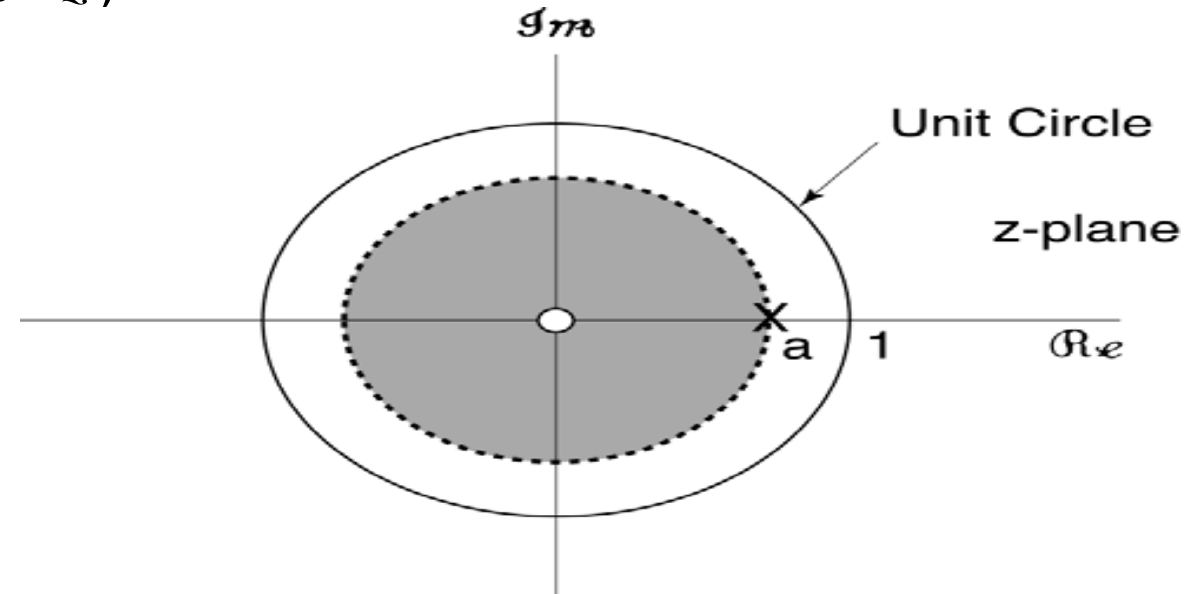
More on ROC

- **Example:** $x[n] = -a^n u[-n - 1]$ (left-sided signal)

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} (-a^n u[-n - 1]) z^{-n} = - \sum_{n=-\infty}^{-1} (a^{-1} z)^n \\ &= 1 - \frac{1 - (a^{-1} z)^{\infty}}{1 - a^{-1} z} = ? \end{aligned}$$

if: $|a^{-1} z| < 1$, or, $|z| < |a|$

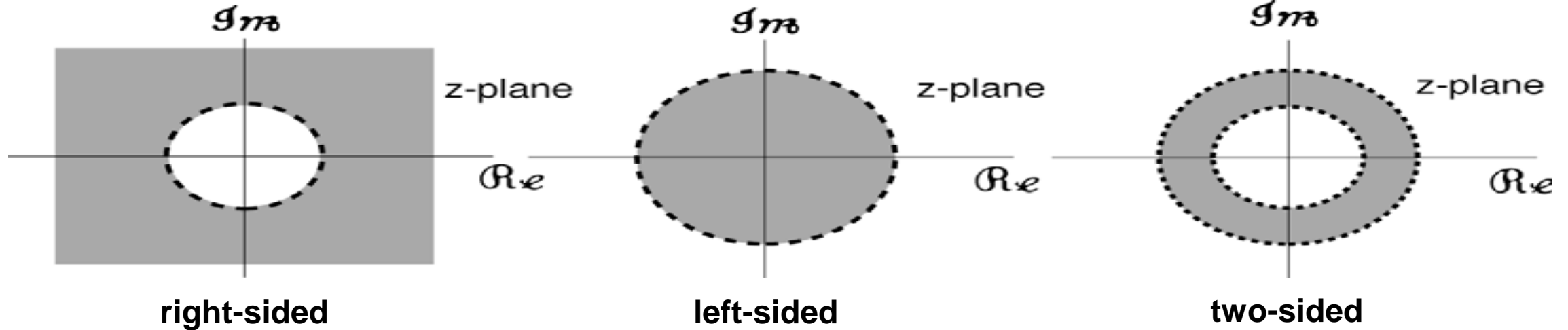
$$\begin{aligned} X(z) &= 1 - \frac{1}{1 - a^{-1} z} = \frac{1 - a^{-1} z}{1 - a^{-1} z} - \frac{1}{1 - a^{-1} z} \\ &= \frac{-a^{-1} z}{1 - a^{-1} z} \\ &= \frac{1}{1 - a z^{-1}} \\ &= \frac{z}{z - a} \end{aligned}$$



The z-Transform is the same, but the region of convergence is different.

Properties of the ROC (Cont.)

- If $x[n]$ is a two-sided sequence, and if $|z| = r_0$ is in the ROC, then the ROC consists of a ring in the z-plane including $|z| = r_0$.

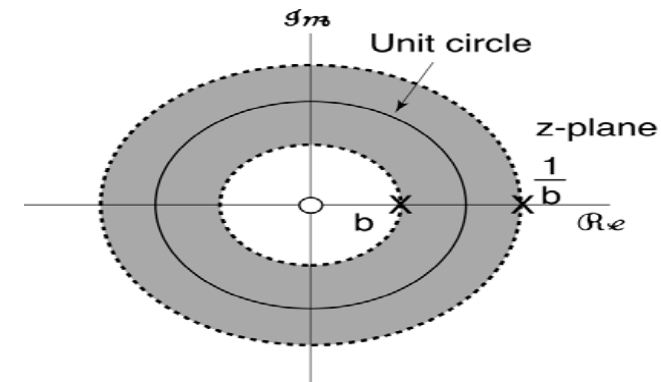
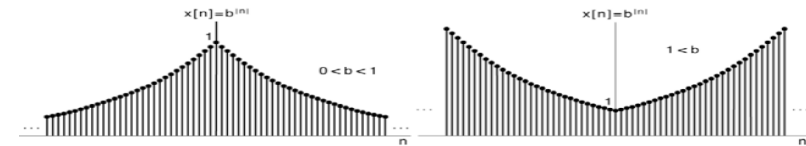


- Example:** $x[n] = b^{|n|} \quad b > 0$
 $x[n] = b^n u[n] + b^{-n} u[-n]$

$$b^n u[n] \iff \frac{1}{1 - bz^{-1}} \quad |z| > b$$

$$b^{-n} u[-n-1] \iff \frac{-1}{1 - b^{-1}z^{-1}} \quad |z| < \frac{1}{b}$$

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1}z^{-1}} \quad \frac{1}{b} < |z| < b$$



Example: Determine the z-transform of the following signals

(a) $x[n] = [1, 2, 5, 7, 0, 1]$

Solution: $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$,

ROC: entire z plane except $z = 0$

(b) $y[n] = [1, 2, 5, 7, 0, 1]$

Solution: $Y(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$

ROC: entire z-plane except $z = 0$ and $z = \infty$.

(c) $z[n] = [0, 0, 1, 2, 5, 7, 0, 1]$

Solution: $z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: all z except $z=0$



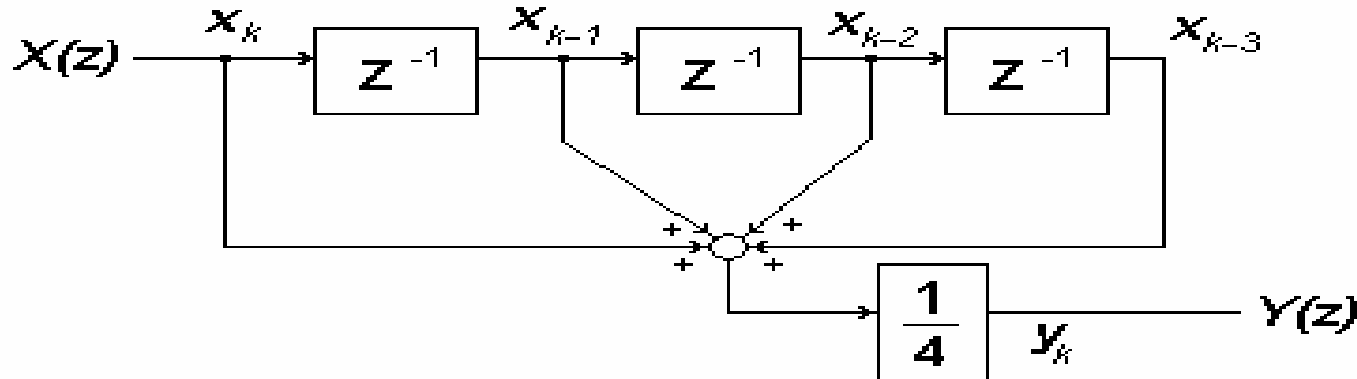
Block Diagram Manipulation

Example 1: Running Average Algorithm

$$y_k = \frac{x_k + x_{k-1} + x_{k-2} + x_{k-3}}{4} \quad (\text{Non-Recursive})$$

$$Y(z) = X(z) \frac{1 + z^{-1} + z^{-2} + z^{-3}}{4} = X(z) \frac{z^3 + z^2 + z + 1}{4z^4} \quad \text{Z Transform}$$

Block Diagram



Transfer Function

$$\frac{Y}{X}(z) = \frac{z^3 + z^2 + z + 1}{4z^4}$$

Note: Each $[Z^{-1}]$ block can be thought of as a memory cell, storing the previously applied value.

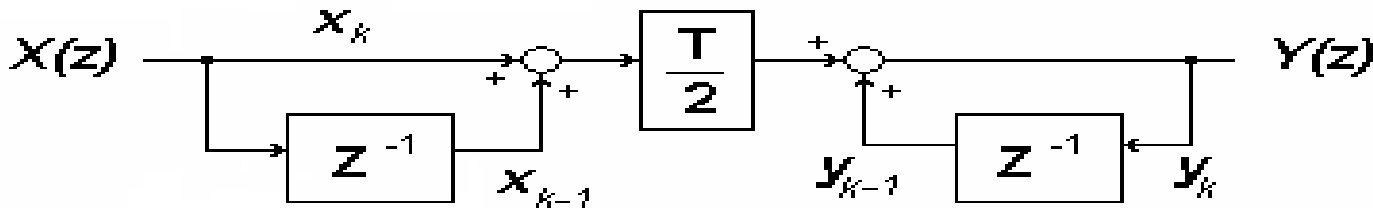
Example 2: Trapezoidal Integrator

$$y_k = y_{k-1} + [x_k + x_{k-1}] \frac{T}{2} \quad (\text{Recursive})$$

$$Y(z) = z^{-1}Y(z) + [X(z) + z^{-1}X(z)] \frac{T}{2} \quad \text{Z Transform}$$

$$Y(z) = X(z) \left[\frac{1 + z^{-1}}{1 - z^{-1}} \right] \frac{T}{2} = X(z) \left[\frac{z + 1}{z - 1} \right] \frac{T}{2}$$

Block Diagram

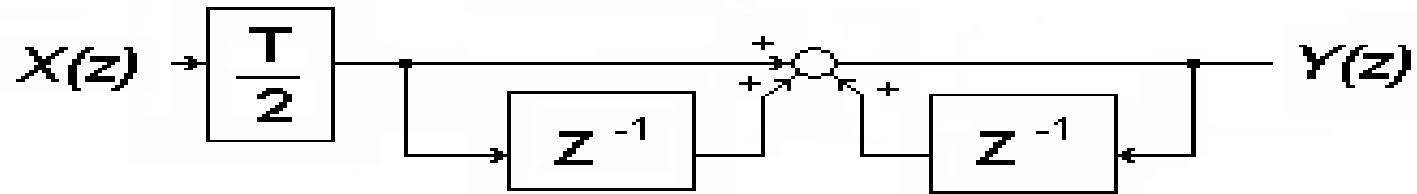
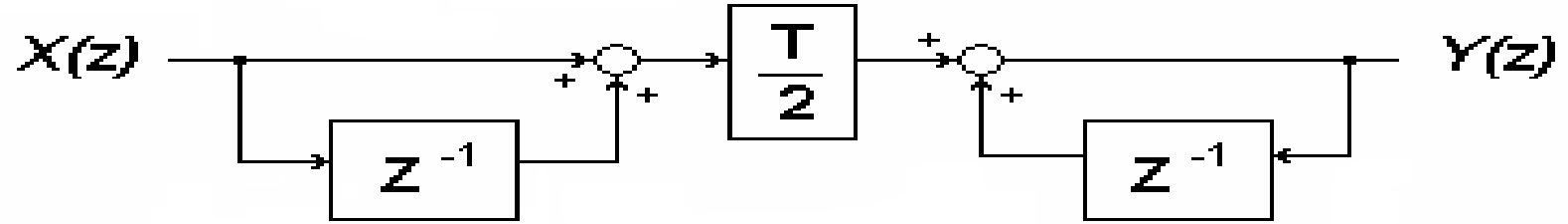


Transfer Function

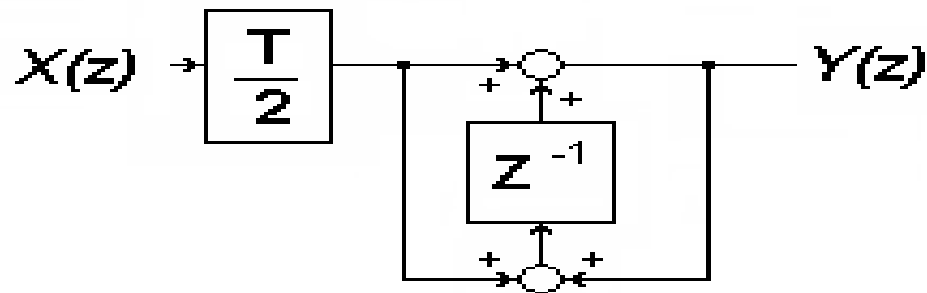
$$\frac{Y(z)}{X(z)} = \frac{T}{2} \left[\frac{z + 1}{z - 1} \right]$$

Ex. 2 (cont) Block Diagram Manipulation

Intuitive Structure

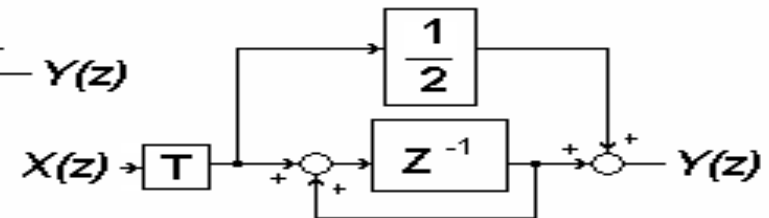
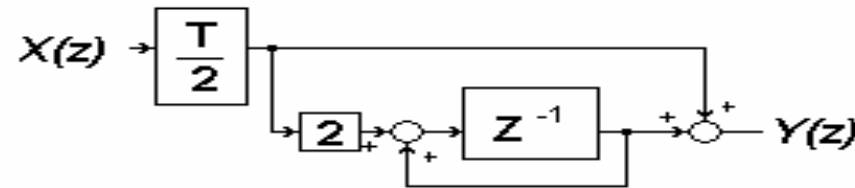
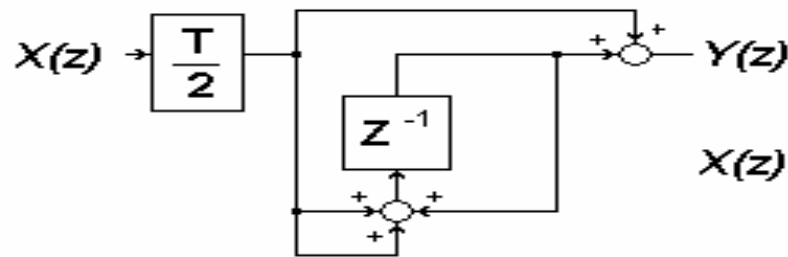
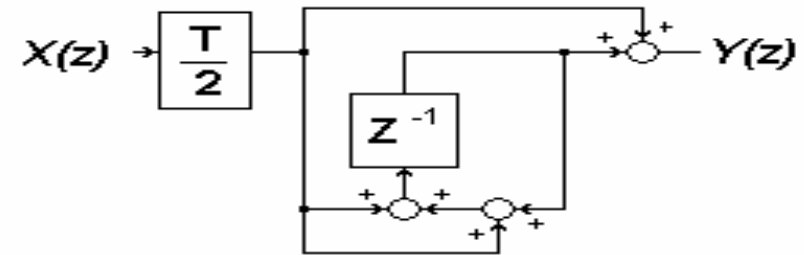
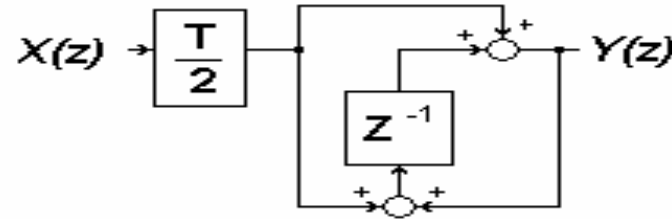
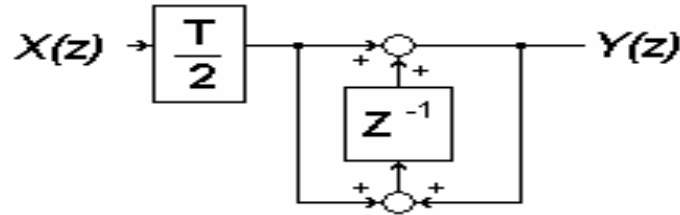


Equivalent Structure



Explicit representation of x_{k-1} and y_{k-1} has been lost, but memory element usage has been reduced from two to one.

Ex. 2 (cont) More Block Diagram Manipulation



$$\frac{Y(z)}{X(z)} = \frac{T}{2} \left[\frac{z+1}{z-1} \right]$$

Note that the final form is equivalent to a rectangular integrator with an additive forward path. In a PI compensator, this path can be absorbed by the proportional term, so there is no advantage to be gained by implementing a trapezoidal integrator.



Inverse Z - Transform

Polinom Bölümü ve Genişletme Metodu

Burada, pay ve payda z^n, z^{n-1}, z^{n-2} veya z^{-1}, z^{-2}, z^{-3} şeklinde yazılarak, uzun bölme işlemi yapılır ve elde edilen bölme sonucunun ters z dönüşümü alınır.

Inverse z Transform Example

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2}, \quad 0.5 < |z| < 2$$

Right-sided signals have ROC's that are outside a circle and left-sided signals have ROC's that are inside a circle. Using

$$\alpha^n u[n] \xleftrightarrow{z} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}, \quad |z| < |\alpha|$$

We get

$$(0.5)^n u[n] + (-2)^n u[-n-1] \xleftrightarrow{z} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2}, \quad 0.5 < |z| < 2$$

Inverse z Transform Example

Find the inverse z transform of

$$X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n u[n] \xleftrightarrow{z} \frac{z}{z - \alpha} = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

We get

$$\left[(0.5)^n - (-2)^n \right] u[n] \xleftrightarrow{z} X(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| > 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathbf{X}(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n u[-n - 1] \xleftrightarrow{z} \frac{z}{z - \alpha} = \frac{1}{1 - \alpha z^{-1}}, \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n \right] u[-n - 1] \xleftrightarrow{z} \mathbf{X}(z) = \frac{z}{z - 0.5} - \frac{z}{z + 2}, \quad |z| < 0.5$$

An Example – Complete Solution

$$U(z) = \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} \quad \longrightarrow \quad U(z) = c_0 + \frac{c_1}{z-2} + \frac{c_2}{z-4}$$

$$c_0 = \lim_{z \rightarrow \infty} U(z) = \lim_{z \rightarrow \infty} \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} = 3$$

$$\begin{aligned} U_2(z) &= (z-2) \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} \\ &= \frac{3z^2 - 14z + 14}{z-4} \end{aligned}$$

$$c_1 = U_2(2) = \frac{3 \cdot 2^2 - 14 \cdot 2 + 14}{2-4} = 1$$

$$\begin{aligned} U_3(z) &= (z-4) \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} \\ &= \frac{3z^2 - 14z + 14}{z-2} \end{aligned}$$

$$c_2 = U_3(4) = \frac{3 \cdot 4^2 - 14 \cdot 4 + 14}{4-2} = 3$$

$$U(z) = 3 + \frac{1}{z-2} + \frac{3}{z-4}$$

$$u(k) = \begin{cases} 3, & k = 0 \\ 2^{k-1} + 3 \cdot 4^{k-1}, & k > 0 \end{cases}$$



Solving Difference Equations

Solving Difference Equations

The unilateral z transform is well suited to solving difference equations with initial conditions. For example,

$$y[n+2] - \frac{3}{2}y[n+1] + \frac{1}{2}y[n] = (1/4)^n, \text{ for } n \geq 0$$

$$y[0] = 10 \text{ and } y[1] = 4$$

z transforming both sides,

$$z^2 [Y(z) - y[0] - z^{-1}y[1]] - \frac{3}{2}z[Y(z) - y[0]] + \frac{1}{2}Y(z) = \frac{z}{z - 1/4}$$

the initial conditions are called for systematically.

Solving Difference Equations

Applying initial conditions and solving,

$$Y(z) = z \left(\frac{16/3}{z-1/4} + \frac{4}{z-1/2} + \frac{2/3}{z-1} \right)$$

and

$$y[n] = \left[\frac{16}{3} \left(\frac{1}{4} \right)^n + 4 \left(\frac{1}{2} \right)^n + \frac{2}{3} \right] u[n]$$

This solution satisfies the difference equation and the initial conditions.



Transfer Function

z-Transform Properties

An LTI system has a transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z - 1/2}{z^2 - z + 2/9}, \quad |z| > 2/3$$

Using the time-shifting property of the z transform draw a block diagram realization of the system.

$$Y(z)(z^2 - z + 2/9) = X(z)(z - 1/2)$$

$$z^2 Y(z) = zX(z) - (1/2)X(z) + zY(z) - (2/9)Y(z)$$

$$Y(z) = z^{-1}X(z) - (1/2)z^{-2}X(z) + z^{-1}Y(z) - (2/9)z^{-2}Y(z)$$



Matlab

Matlab

- $ztrans(f)$ finds the Z-Transform of f . By default, the independent variable is n and the transformation variable is z . If f does not contain n , $ztrans$ uses $symvar$.

Mathematical Operation	MATLAB Command
$\lim_{x \rightarrow 0} f(x)$	<code>limit(f)</code>
$\lim_{x \rightarrow a} f(x)$	<code>limit(f, x, a)</code> or <code>limit(f, a)</code>
$\lim_{x \rightarrow a^-} f(x)$	<code>limit(f, x, a, 'left')</code>
$\lim_{x \rightarrow a^+} f(x)$	<code>limit(f, x, a, 'right')</code>

Matlab - Örnekler

```
clc  
clear all  
close all
```

```
x = [9 2 3 4 5];  
b=0;  
n=length(x);  
y=sym('z');  
for i=1:n  
    b=b+x(i)*y^(1-i);  
end  
display(b);
```

```
clc  
clear all  
close all  
syms z k  
f=0.5.^k;  
a=ztrans(f)  
pretty (a)  
  
a = z/(z - 1/2)
```

$$\frac{z}{z - 1/2}$$

```
clc  
clear all  
close all  
syms z k  
F=z/(z-0.5);  
a=iztrans(F,k)  
pretty(a)
```

$$a = \frac{(1/2)^k}{(1/2)}$$

$$b = 2/z + 3/z^2 + 4/z^3 + 5/z^4 + 9$$

Usage Notes

- These slides were gathered from the presentations published on the internet. I would like to thank who prepared slides and documents.
- Also, these slides are made publicly available on the web for anyone to use
- If you choose to use them, I ask that you alert me of any mistakes which were made and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides.

Sincerely,

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